statement: Let $(f_n)_n \in \mathcal{H}(\Omega)$. By another theorem $f := \lim_{n \to \infty} f_n$ is holomorphic if $(f_n)_n$ converges uniformly to f. The claim is that then $(f_n^{(k)})_n$ converges uniformly to $f^{(k)}$ on Ω.

proof: Let $\Omega_\delta := \{x \in \Omega \mid \overline{D_\delta}(x) \subset \Omega\}$. We will show that $(f_n')_n$ converges uniformly to f' on Ω_δ. Therefore, we first prove:

$$|F'(z_0)| \leq \frac{1}{\delta} \sup_{z \in \Omega} |F(z)| \qquad (1)$$

for a function $F \in \mathcal{H}(\Omega)$, $z_0 \in \Omega_\delta$.

By C.R. ineq. we immediately observe:

$$|F'(z_0)| \leq 1! \ \frac{\sup_{z \in C_\delta} |F(z)|}{\delta} \leq \frac{1}{\delta} \sup_{z \in \Omega} |F(z)|$$

as $\overline{D_\delta}(z_0) \subset \Omega$.

Now we take $F = f_n - f \in \mathcal{H}(\Omega)$ because $f_n, f \in \mathcal{H}(\Omega)$.

We obtain $\forall z_0 \in \Omega_\delta : |f_n'(z_0) - f'(z_0)| \leq \frac{1}{\delta} \sup_{z \in \Omega} |(f_n - f)(z)|$

As $n \to \infty$ $\sup_{z \in \Omega} |(f_n - f)(z)| \to 0$ uniformly according to our prerequisits and hence, $|f_n'(z_0) - f'(z_0)| \to 0$ uniformly on Ω_δ. Now when $\delta \to 0$, $\Omega_\delta \to \Omega$ and therefore, $f_n'(z_0) \to f'(z_0)$ on Ω uniformly. Higher derivatives are obtain by inductively using this argument.

parameter integral for complex functions

statement: Let $f(z) := \int_a^b F(z,s)\, ds$ for $(z,s) \in \Omega \times [a,b]$ for

Ω open in $\mathbb{C}$. If

i) $F(z,s) \in \mathcal{H}(\Omega)$ for each s

ii) $F(z,s) \in C^1(\Omega \times [a,b]$

Then $f(z) \in \mathcal{H}(\Omega)$

proof 1: $\int_T f(z) = \int_T \int_a^b F(z,s)\, ds\, dt \overset{\text{Fubini}}{=} \int_a^b \int_T F(z,s)\, dt\, ds \overset{\substack{F \in \mathcal{H}(\Omega) \\ \text{for fixed } s,}}{\underset{\text{Goursat}}{=}}$

$= \int_a^b 0\, ds = 0$

$\overset{\text{Morera}}{\Longrightarrow} f \in \mathcal{H}(\Omega)$ $\square$

proof 2: Consider $f_n(z) = \frac{1}{n} \sum_{k=1}^{n} F(z, k/n)$ for $[a,b] = [0,1]$.

Claim that $(f_n)_n$ converges uniformly to f and is holomorphic. Obviously, due to property i) $f_n \in \mathcal{H}(\Omega)$.

Now, since $F \in C^1(\Omega \times [0,1])$, $\forall \varepsilon\, \exists \delta : \forall s_1, s_2 \in [0,1] : |s_2 - s_1| < \delta$

$\Rightarrow \sup_{z \in D} |F(z,s_2) - F(z,s_1)| < \varepsilon$ (1)

For $n > \frac{1}{\delta}$ we now obtain:

$|f_n(z) - f(z)| \leq \left|\sum_{k=1}^{n} \int_{\frac{k-1}{n}}^{k/n} F(z, \tfrac{k}{n}) - F(z,s)\, ds\right| \overset{(1)}{\leq}$

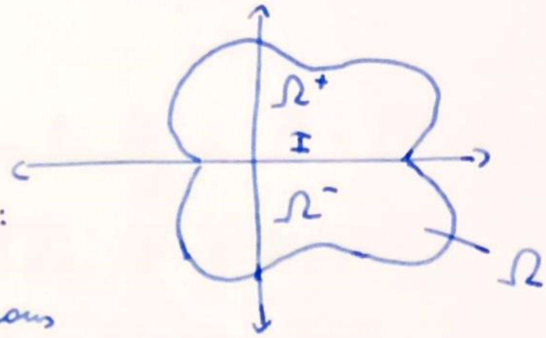
$\leq \left|\sum_{k=1}^{n} \int_{\frac{k-1}{n}}^{k/n} \varepsilon\, ds\right| \leq n \cdot \frac{\varepsilon}{n} \leq \varepsilon$

$\Rightarrow$ uniform convergence of $f_n \to f$.

$\overset{\text{th. uniform conv.}}{\underset{\text{\& holomorphicity } f_n}{\Longrightarrow}} f \in \mathcal{H}(\Omega)$ $\square$

The symmetry principle

statement: Let Ω be symmetric to the real axis, meaning $\forall z \in \Omega$: $\bar{z} \in \Omega$, and open.



Then for $f^+ \in \mathcal{H}(\Omega^+)$,
$f^- \in \mathcal{H}(\Omega^-)$ with $\forall x \in I$:
$f^+(x) = f^-(x)$ after a continuous extension of f^+, f^- to I,

$$f(z) := \begin{cases} f^+(z) & | \; z \in \Omega^+ \\ f^+(z) = f^-(z) & | \; z \in I \\ f^-(z) & | \; z \in \Omega^- \end{cases} \in \mathcal{H}(\Omega).$$

proof: We only need to prove holomorphicity for $z_0 \in I$. Let therefore be $z_0 \in I$, $D = D_r(z_0) : D \subset \Omega$. Then $\int_T f(z)\,dz = 0$ for all triangles, since  I can be split into triangles in the upper and lower half and a gap of 2ε around I. As $\varepsilon \to 0$ by continuity $\int_{T_i} f(z)\,dz = 0$ since T_i is still entirely in the upper/lower half of Ω.
By Morera we conclude the claim.

The Schwarz reflection principle

Statement: Let $f \in \mathcal{H}(\Omega^+)$ with a continuous extension on I s.t. $\forall z \in I : f(z) \in \mathbb{R}$.

Then f can be extended holomorphically on $\Omega = \Omega^+ \cup I \cup \Omega^-$.

proof: Define $F = \overline{f(\bar{z})}$ for $z \in \Omega^-$. Then $F \in \mathcal{H}(\Omega^-)$ since $f \in \mathcal{H}(\Omega^+)$ and $\forall z, z_0 \in \Omega^- : F(z) = \overline{f(\bar{z})} =$

Power ser. ext.
$$= \overline{\sum_{n=0}^{\infty} a_n (\bar{z} - \bar{z}_0)^n} = \sum_{n=0}^{\infty} \bar{a}_n (z - z_0)^n \text{ is analytic}$$

$\Rightarrow$ holomorph. Furthermore F extends to I s.t. $F(z \in I) = \overline{f(\bar{z})} = \overline{f(z)} = f(z) \in \mathbb{R}$. Use the symmetry principle to conclude $(f \cup F) := \begin{cases} f & z \in \Omega^+ \\ f = F & z \in I \\ F & z \in \Omega^- \end{cases}$ $\in \mathcal{H}(\Omega)$

$\square$

Runge's approximation theorem

statement: Let $f \in \mathcal{H}(U)$ for U an neighbourhood of the compact set K. Then f can be approximated uniformly by rational functions with singularities in K^c.
If K^c is connected, f can be approximated uniformly by polynomials.

proof: [We find finitely many segments γ_i (using a grid) s.t. $\forall z \in K: f(z) = \sum_{n=1}^{N} \frac{1}{2i\pi} \int_{\gamma_n} \frac{f(\xi)}{\xi - z} d\xi$ with $\gamma_i \in \Omega \setminus K$.]

Now let $\gamma \in \Omega \setminus K$. We consider $\int_\gamma \frac{f(\xi)}{\xi - z} d\xi =$
$\int_0^1 \frac{f(\gamma(t))}{\gamma(t) - z} \gamma'(t) dt$, was nach dem Riemann-Summen aus komplexen Parameterintegral als rationale Fkt. approximiert werden kann, da die Singularitäten auf γ liegen, $F(z,t) = \int_0^1 \frac{f(\gamma(t))}{\gamma(t)-z} \gamma'(t) dt$ auf K stetig ist und in der Variable z holomorph.

Now, if K^c is connected we can choose z_1 outside an arbitrarily large disc, since K is bounded. Now we notice that $\frac{1}{z_1 - z_0} = \frac{1}{z_1} \frac{1}{1 - \frac{z_0}{z_1}} = \frac{1}{z_1} \sum_{n=0}^{\infty} \left(\frac{z_0}{z_1}\right)^n$

Taking $(w_j)_j$ on γ $(z_0 \to z_1)$ we obtain that $\frac{1}{z - z_0}$ can be uniformly approximated by polynomials $\sum_{n=0}^{\infty} \frac{z_0^n}{w_j^{n+1}}$ and we conclude the theorem.

Neighbourhood of zeros

statement: Let $f \in \mathcal{H}(\Omega)$, $f(z_0) = 0$. Then $f \neq 0 \Rightarrow \exists U \subset \Omega$:
$\forall z \in U: f(z) \neq 0 \wedge f(z) = (z - z_0)^n g(z)$ with $g(z_0) \neq 0$.

proof: i) Due to the ACP we know that if $\forall U$ neighbourhood of z_0 $\exists z: f(z) = 0$, we have a sequence converging to z_0 with $(f(z_n))_n = 0$, therefore find a neighbourhood of z_0 U_0 with $\forall w \in U_0 : f(w) = 0$ (by ACP) and due to the uniqueness of analytic continuation we find $f = 0$ on Ω, a contradiction to $f \neq 0$. So $\exists U \subset \Omega : \forall z \in U : f(z) \neq 0$.

ii) Because $f \in \mathcal{H}(\Omega)$, f is analytic and since $f \neq 0$:
$\exists n : a_n$ in the Taylorexpansion $\neq 0$: $f(z) = \sum_{i=0}^{\infty} a_i (z - z_0)^n =$
$= \sum_{i=n}^{\infty} a_i (z - z_0)^n = (z - z_0)^n (\underbrace{a_n}_{\neq 0} + a_{n+1}(z - z_0)^1 + \ldots + \ldots)$ and

therefore $f(z) \neq 0$ for z close to z_0. Furthermore, the order of vanishing n is unique since if $f(z) = (z - z_0)^m h(m)$, then $a_m = 0$ for $m < n$ and therefore, $h(z_0) = 0$. If $m > n$, then $g(z_0) = (z - z_0)^{m-n} h(z_0) = 0$ ↯ $\Rightarrow g(z) = h(z)$ and n is unique.

The residue formula

statement: Let f be holomorphic on $\Omega \setminus \{z_0\}$ with a pole at z_0.
Then for any disc $\overline{D_r}(z) \subset \Omega$ with $z_0 \in D_r(z)$ and its boundary C we get: $\int_C f(z)\,dz = 2\pi i \, res_{z_0}(f)$.

proof: We use the keyhole contour, obtain $\int_C f(z)\,dz = \int_{C_\varepsilon} f(z)\,dz =$

$$= \int_{C_\varepsilon} \sum_{k=1}^{n} \frac{a_{-k}}{(z-z_0)^k} + G(z)\,dz \overset{*)}{=} 2\pi i \, a_{-1} = 2\pi i \, res_{z_0}(f).$$

*) i) $\forall k > 1: \int_{C_\varepsilon} \frac{a_{-k}}{(z-z_0)^k} \overset{C.R.eq}{=} 2\pi i \, (a_{-k})^{(k-1)} = 0$

ii) $\int_{C_\varepsilon} G(z)\,dz \overset{G \in \mathcal{H}(D)}{=} 0$

iii) $\int_{C_\varepsilon} \frac{a_{-1}}{(z-z_0)} \overset{C.R.eq.}{=} 2\pi i \, a_{-1}$

remark: This formula can be extended to finitely many residues
$\left[\int_C f(z)\,dz = 2\pi i \sum_{i=1}^{n} res_{z_i}(f) \text{ for } z_i \text{ residues} \in D \right]$
in arbitrary toy contours.

$\Rightarrow$ The residue formula: $\int_\gamma f(z)\,dz = 2\pi i \sum_{i=1}^{N} res_{z_i}(f)$

Riemann's theorem on removable singularities

statement: Let $f \in \mathcal{H}(D^\cdot)$ be bounded on $D^\cdot$. Then f has a removable singularity at z_0, meaning $\exists g \in \mathcal{H}(D): \forall z \in D^\cdot$: $g(z) = f(z)$.

proof: Define $\tilde{f}(z) = \begin{cases} 0 & z = z_0 \\ (z-z_0)^2 f(z) \end{cases}$. Since $\frac{\tilde{f}(z_0+h) - \tilde{f}(z_0)}{h} =$

~~$((z_0+h-z_0)^2 - (z_0-z_0)^2)$~~

$= \frac{h^2 f(z_0+h)}{h} = h f(z_0+h) \underset{f \text{ bounded}}{\overset{h \to 0}{=}} 0$, we conclude

$\tilde{f} \in \mathcal{H}(D)$. Since $\tilde{f}(z_0) = \tilde{f}'(z_0) = 0$, $\mathrm{ord}_{z_0}(\tilde{f}) \geq 2$

FI: $\mathrm{ord}_{z_0}(\tilde{f}) = \infty \Rightarrow \tilde{f} = 0 \Rightarrow f = 0 \Rightarrow g = 0$

FII: $\mathrm{ord}_{z_0}(\tilde{f}) = k \geq 2$. Define $\tilde{f}_1$: $\tilde{f}(z) = (z-z_0)^k \tilde{f}_1(z)$

$\forall z \in D^\cdot$ $\Rightarrow f(z) = (z-z_0)^{k-2} \tilde{f}_1(z)$ with $\tilde{f}_1 \in \mathcal{H}(D)$.

$\Rightarrow g(z) = (z-z_0)^{k-2} \tilde{f}_1(z)$ to expand f on D.

Corollary: condition for poles

statement: $f \in \mathcal{H}(U^\cdot_{z_0})$ has a pole at $z_0 \Leftrightarrow |f(z)| \to \infty$ for $z \to z_0$.

proof: If f has a pole at z_0, then by def. $1/f$ has a zero at z_0. Therefore, $\left|\frac{1}{f(z)}\right| \to 0 \Rightarrow |f(z)| \to \infty$ for $z \to z_0$.

If $|f(z)| \to \infty$ for $z \to z_0$, $\frac{1}{f}$ has a zero at z_0 and is holomorphic on U_{z_0}. Therefore, z_0 is a pole of f (per def.)

Casorati - Weierstrass - Theorem

statement: Let $f \in \mathcal{H}(D_r^\circ(z_0))$. If f has an essential singularity at z_0, then the image of f is dense in the complex plane.

proof: Suppose the contrary holds. Then we find $w \in \mathbb{C}, \delta \in \mathbb{R}$:

$\forall z \in D_r^\circ(z_0)$: $|f(z) - w| > \delta$.

Define $g(z) = \dfrac{1}{f(z) - w}$ which is bounded by $1/\delta$ on $D_r^\circ(z_0)$.

If $g(z_0) = \lim\limits_{z \to z_0} \dfrac{1}{f(z)-w} \neq 0$, g has a removable singularity at z_0 and hence, so does f. ↯

If $g(z_0) = 0$, has a pole at

Since $\exists r'$: g is bounded on $D_{r'}^\circ(z_0)$, g has a removable singularity at z_0.

Hence, $f(z) = \dfrac{1}{g(z)} + w$ has a

i) removable singularity at z_0 if $g(z) \neq 0$

ii) pole at z_0 if $g(z) = 0$

$\Big\}$ ↯ z_0 is essential singularity

remark: Great Picard's theorem: If $f \in \mathcal{H}(D_r^\circ(z_0))$ has an essential singularity at z_0, it takes on every complex number infinitely often with maximum one exception.

meromorphic and rational functions

statement: The meromorphic functions in the extended complex plane are rational.

proof: Since f has either a pole or is holomorphic at infinity, $f(1/z)$ has a deleted neighbourhood on which it is holomorphic. Therefore, f only has finitely many poles $z_1, \dots, z_N, (z_\infty)$. For each z_i we can write $f(z) = f_i(z) + g_i(z)$ close to z_i where f_i is the principal part of f and $g_i \in \mathcal{H}(D_r(z_i))$. Now define $H(z) = f(z) - \sum_{i=1}^{N} f_i(z) - f_\infty(z)$

Then H is bounded since in a neighbourhood of a pole z_i we always subtracted the principal part. Therefore, H is constant and as $f_i(z)$ are of the form

$$\sum_{j=1}^{M_i} \frac{a_{i,j}}{(z-z_0)^j}$$ and $f_\infty(z)$ = principal part of $f(1/z)$

has the form $\sum_{j=1}^{M_\infty} \frac{a_{\infty,j}}{z^j}$, we conclude that

$$f(z) = \underbrace{H(z)}_{\text{const}} + \underbrace{f_\infty(z)}_{\text{polynomial in } \frac{1}{z}} + \underbrace{\sum_{i=1}^{N} f_i(z)}_{\text{polynomials in } \frac{1}{z-z_i}}$$ is rational. □

Lemma: logarithmic derivative

statement: Let $f \in \mathcal{M}(U)$. Then the logarithmic derivative of f: $\frac{f'}{f}$ has poles of order 1 wherever f has poles or zeros and its residues at these z is equal to $\mathrm{ord}_z(f)$.

proof: Define $U_f := U \setminus \{z_i \mid \mathrm{ord}_{z_i}(f) \neq 0\}$.

Obviously $\frac{f'}{f} \in \mathcal{H}(U_f)$.

For z_0 pole of f we observe: $f(z) = \frac{\tilde{f}(z)}{(z-z_0)^j}$ for $\mathrm{ord}_{z_0}(f) = j$.

So $$\frac{f'}{f} = \frac{\frac{\tilde{f}'(z)}{(z-z_0)^j} + \frac{\tilde{f}(z)\cdot(-j)}{(z-z_0)^{j+1}}}{\tilde{f}(z)\frac{1}{(z-z_0)^j}} = \frac{\tilde{f}'(z)}{\tilde{f}(z)} + \frac{-j}{z-z_0}$$

has a pole of order 1 at z_0 since $\tilde{f} \in \mathcal{H}(U)$.

Furthermore, $f'(z) = \frac{\tilde{f}'(z)(z-z_0) + \tilde{f}(z)(-j)}{(z-z_0)^{j+1}}$ has a pole of order $j+1$ because $\tilde{f}(z_0) \neq 0$, $j \neq 0$.

Moreover, for z_0 zero of f we observe $(f(z) = (z-z_0)^k h(z))$:

$$\frac{f'}{f} = \frac{k\cdot(z-z_0)^{k-1}h(z) + (z-z_0)^k h'(z)}{(z-z_0)^k h(z)} = \frac{k}{z-z_0} + \frac{h'}{h}$$

and therefore, f'/f has a pole of order 1 at z_0.

For $\mathrm{res}_{z_0}(f'/f)$ we observe 1) $\mathrm{res}_{z_0}(f'/f) = \lim_{z\to z_0}(z-z_0)\frac{\tilde{f}'}{\tilde{f}} - j = -j$ bzw. $\mathrm{res}_{z_0}(f) = \lim_{z\to z_0}(z-z_0)\frac{h'}{h} + k = k = \mathrm{ord}_{z_0}(f)$ $\square$

The argument principle

statement: Let $f \in \mathcal{M}(U)$, $\gamma : [0,1] \to U$, s.t. $\nexists z$:
$\text{ord}_z(f) \neq 0 \wedge \exists x : \gamma(x) = z$.

Then $$\frac{1}{2\pi i} \int_\gamma \frac{f'(w)}{f(w)}\, dw = \sum_{\substack{z \text{ zero} \\ \text{of } f}} \text{ord}_z(f) + \sum_{\substack{z \text{ pole} \\ \text{of } f}} \text{ord}_z(f) =$$

$$= \sum_{\substack{z: \\ \text{ord}_z(f) \neq 0}} \text{ord}_z(f) = \sum_{z \in D} \text{ord}_z(f) = \#\,\text{zeros} - \#\,\text{poles}$$

with their multiplicities.

proof: logarithmic derivative & residue theorem.

corollary: This holds for arbitrary toy contours.

corollary (Argument principle 1): Rouché's theorem

statement: Let $D_r(z_0) \subseteq U \subseteq \mathbb{C}$ and $f, g \in \mathcal{H}(U)$.
If $\forall z \in C$: $|f(z)| > |g(z)|$, then
f and $f+g$ have the same number of zeros
inside $C :=$ boundary of $D_r(z_0)$

proof: i) If f has a zero at z_0, then since $|f(z_0)| > |g(z_0)|$,
g has a zero at z_0 $\Rightarrow$ $f+g$ has a zero at z_0

ii) If $f+g$ has a zero at z_0, the either $f(z_0) = -g(z_0)$
or $f(z_0) = g(z_0) = 0$. But since $|f(z_0)| > |g(z_0)|$
only the second case is possible $\Rightarrow$ f has a zero at z_0.

$\square$

The open image theorem

statement: If U open and connected, $f \in \mathcal{H}(U)$ non-const.,
then $f(U) \subseteq \mathbb{C}$ open. (also: $V \subseteq U$ open $\Rightarrow f(V)$ open)

proof: Let $f(z_0) = w_0$. We want to show that $\exists r : D_r(w_0) \subset f(V)$ for
V open, $\overline{D_\delta(z_0)} \subset V$. Define $\tilde{f}(z) = f(z) - w_1 = \underbrace{f(z) - w_0}_{F} +$
$\underbrace{w_0 - w_1}_{G}$ where we'd like to show $\exists z : f(z) = w_1$.

We now choose δ, s.t. for $|z - z_0| \leq \delta : z \in U \wedge$
$z \in C_\delta \Rightarrow f(z) \neq w_0$. Next we choose ε, s.t. $f(z) - w_0 \geq \varepsilon$ for
$z \in C_\delta :=$ boundary of $D_\delta(z_0)$. But then we find $\forall w_1$:
$|w_1 - w_0| < \varepsilon$: $|F(z)| \geq \varepsilon > |G(z)|$

$\overset{\text{Rouché's th.}}{\Longrightarrow}$ $\tilde{f}(z) := f(z) - w_1 = F + G$ has a zero since F has
one $\Rightarrow \forall w_1 \in D_\varepsilon(w_0)$: $\exists z : f(z) = w_1 \Rightarrow f(V)$ is
open. $\square$

Maximum modulus principle

statement: If f is non-const, U open, connected, $f \in \mathcal{H}(U)$,

then $\nexists z_0 \in U : \forall z \in U : |f(z)| \leq |f(z_0)|$.

In particular, the maximum of $\overline{U}$ is obtain at its boundary : $\max\limits_{z \in \overline{U}} |f(z)| = \max\limits_{z \in \overline{U} - U} |f(z)|$

$\uparrow$ exists since $\overline{U}$ compact.

proof: I) Assume $\exists z_0 \in U : \forall z \in U : |f(z)| \leq |f(z_0)|$

Then $f(D_\delta(z_0)) \subseteq \{w \mid f(w) \leq f(z_0)\}$ and

$\nexists \varepsilon : D_\varepsilon(f(z_0)) \subseteq \{w \mid f(w) \leq f(z_0)\}$ $\Rightarrow$ $\lightning$ open mapping th.

II) Since $\overline{U}$ is bounded and closed, it's compact. Therefore f attains a maximum in $\overline{U}$, denoted by z_0, and since z_0 cannot lie in U due to I), we get $z_0 \in \overline{U} \setminus U$. $\square$

3. Let $V = \{x \mid f(x) = c\} \subset U$. Then V is open, since $\forall x \in V\ \exists r > 0$:

$\forall z \in D_r(x): f(z) = c$ since $f' = 0 \Rightarrow \forall m \geq 1: f^{(m)} = 0$

$$\Rightarrow f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x)}{k!}(z-x) = f(x) \quad \text{in } D_r(x).$$

And V is closed aswell since $V^c = \{y \mid f(y) \neq c\}$ is open due to the fact that if $y_0 \in V^c \nexists y \in D_r(y_0) : f(y) = c$ $\forall r$

$\overset{f \in \mathcal{H}(U)}{\Longrightarrow} f(y_0) = c$ $\lightning$.

$\overset{V \text{ non-empty}}{\underset{U \text{ connected}}{\Longrightarrow}} V = U$ $\square$

Convergence of an infinite product

statement: If $\sum_{n\in\mathbb{N}} |a_n|$ converges for a sequence $(a_n)_n$, the both infinite products $\prod_{n\in\mathbb{N}} (1+a_n)$ and $\prod_{n\in\mathbb{N}} (1+|a_n|)$ converge. We speak of absolute convergence.

proof: i) For $\prod_n (1+|a_n|)$ we notice that

$$\prod_n (1+|a_n|) \leq \prod_n \exp(|a_n|) = \exp\Big(\sum_n |a_n|\Big)$$

and therefore, it converges.

ii) For $\prod_n (1+a_n)$ we consider

$$\Big|\prod_{n=1}^{N+1}(1+a_n) - \prod_{n=1}^{N}(1+a_n)\Big| = \Big|\prod_{n=1}^{N}(1+a_n)\cdot a_{N+1}\Big| =$$

$$= |a_{N+1}|\cdot\Big|\prod_{n=1}^{N}(1+a_n)\Big| \leq |a_{N+1}|\cdot\prod_{n=1}^{N}(1+|a_n|) =$$

$$\prod_{n=1}^{N+1}(1+|a_n|) - \prod_{n=1}^{N}(1+|a_n|).$$

Hence, $\prod_n (1+a_n)$ is Cauchy, since $\prod_n (1+|a_n|)$ converges, and as a result converges, too

□

Lemma: infinite product equals zero

statement: A converging infinite product $\prod_{n\in\mathbb{N}} 1+a_n = 0 \Leftrightarrow \exists n: a_n = -1$

proof: For $a_n \neq -1 \wedge \sum_n |a_n|$ converges, $\sum_n \left|\frac{a_n}{1+a_n}\right|$ converges and therefore,

$$\prod_{n=1}^{N}\Big(1-\frac{a_n}{1+a_n}\Big) = \prod_{n=1}^{N}\frac{1}{1+a_n} = \frac{1}{\prod_{n=1}^{N}(1+a_n)} \text{ conv.} \Rightarrow \prod_n (1+a_n) \neq 0$$

□

Convergence of the infinite product of functions

statement: Let $a_n \in \mathcal{H}(U)$. If $\sum_n a_n(z)$ converges uniformly on compact subsets of U, then $\prod_n (1 + a_n(z)) = f(z) \in \mathcal{H}(U)$

proof: We again notice $\prod_n (1 + |a_n(z)|) \leq \prod_n \exp(|a_n(z)|) =$
$= \exp(\sum_n |a_n(z)|)$ and $|f(z) - \prod_{n=1}^{N} (1 + a_n(z))| \leq$
$\leq g(z) - \prod_{n=1}^{N} (1 + |a_n(z)|)$ with $g(z) = \prod_n (1 + a_n(z))$
on K compact $\subseteq U$. Therefore, $f \in \mathcal{H}(K)$ and by the convergence theorem we conclude $f \in \mathcal{H}(U)$. □

$\eta_{Ded} \in \mathcal{H}(\mathbb{H})$

proof: $\eta_{Ded}(z) = e^{\frac{i\pi z}{12}} \prod_{n \geq 1} 1 - e^{2i\pi z n}$, so it suffices to prove that $\prod_{n \geq 1} 1 - e^{2i\pi n z} \in \mathcal{H}(\mathbb{H})$.

We define $a_n(z) := e^{2i\pi z n}$, notice $|a_n(z)| = e^{-2\pi \operatorname{Im}(z) n} =$
$\alpha(z)^n$ for $\alpha(z) := e^{-2\pi \operatorname{Im}(z)}$. We estimate
$|\alpha(z)| \leq e^{-2\pi y_0}$ for $y_0 \leq \operatorname{Im}(z)$ and notice, that
$|a_n(z)| = \alpha(z)^n \leq (e^{-2\pi y_0})^n$ converge uniformly for $\operatorname{Im}(z) > 0$.
(especially: $e^{-2\pi \operatorname{Im}(z)} \in]0, 1[$)

$\Rightarrow \eta \in \mathcal{H}(\mathbb{H})$ □

Theorem: The Euler formula for ζ defines a hol. Fct.

statement: $\prod_{p \text{ prim}} (1-p^{-s})^{-1} \in \mathcal{H}(U_1 := \{z \in \mathbb{C} \mid Re(z) > 1\})$

proof: i) Bring $\prod_{p \text{ prim}} (1-p^{-s})^{-1}$ to the form $\prod_{p \text{ prim}} 1 + a_p(s)$:

Define $a_p = \frac{p^{-s}}{1-p^{-s}}$. Then $\frac{1}{1-p^{-s}} = 1 + a_p(s)$.

ii) Show $\sum_{p \text{ prim}} a_p(s)$ conv. uniformly on compact sets.

We first notice that $\frac{1}{p^s} \neq 1$ since

$\left|\frac{1}{p^s}\right| \leq \left|\frac{1}{2^s}\right| \overset{s=\sigma+it}{=} \frac{1}{2^\sigma} < 1$ since $s \in U_1 \Rightarrow \sigma > 1$.

Then define compact sets $K_\delta := \{z \in \mathbb{C} \mid Re(z) \geq 1+\delta\}$

We find $\left|\sum_p a_p(s)\right| \leq \frac{1}{1-2^\sigma} \sum_p p^{-\sigma} \leq \frac{1}{1-2^\sigma} \underbrace{\sum_n n^{-(1+\delta)}}_{\text{converges}}$

$\overset{\text{conv. for inf. pr. for fct.}}{\Longrightarrow} \prod_p (1-p^{-s})^{-1} \in \mathcal{H}(U_1)$

Theorem: Euler's formula for ζ coincides with ζ

statement: On $U_1 = \{z \in \mathbb{C} \mid Re(z) > 1\}$ Euler's f. for ζ coincides with ζ

proof: i) Notice: $P_x = \prod_{p \leq x} (1-p^{-s})^{-1} \overset{\text{geo. R.}}{=} \prod_{p \leq x} (1 + p^{-\sigma} + \dots + p^{-n\sigma} + \dots)$

$= \sum_{n \in S_x} n^{-\sigma}$ where $S_x := \{n \in \mathbb{N} \mid n$ has only prime factors $p_i \leq x\}$.

ii) Use $\sum_{\substack{n \in S_x \\ n \leq x}} n^{-\sigma} \leq P_x(\sigma) \leq \sum_{n \in S_x} n^{-\sigma} \leq \sum_{n \geq 1} n^{-\sigma}$

to observe $P_x(\sigma) = \zeta(\sigma)$ and use analytic continuation $\square$